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## ON THE FIRST DIGITS OF THE FIBONACCI NUMBERS AND THEIR EULER FUNCTION

FLORIAN LUCA — PANTELIMON STĂNICĂ

ABSTRACT. Here, we show that given any two finite strings of base  $b$  digits, say  $s_1$  and  $s_2$ , there are infinitely many Fibonacci numbers  $F_n$  such that the base  $b$  representation of  $F_n$  starts with  $s_1$  and the base  $b$  representation of  $\phi(F_n)$  starts with  $s_2$ .

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### 1. Introduction

Let  $b \geq 2$  be and integer. Let  $s_1 = \overline{c_1 \cdots c_{k(b)}}$  be a positive integer  $s_1$  written in base  $b$ . Washington [4] proved that there exist infinitely many Fibonacci numbers  $F_n$  whose base  $b$  representation starts with  $s_1$ . In fact, the first digits of the Fibonacci sequence obey Benford's law in that the proportion of the positive integers  $n$  such that  $F_n$  starts with  $s_1$  is precisely  $\log((s_1 + 1)/s_1)/\log b$ . Here, we take this one step further. Let  $\phi(m)$  be the Euler function of the positive integer  $m$ . We put  $s_2 = \overline{d_1 \cdots d_{\ell(b)}}$  for some other positive integer written in base  $b$  and prove the following theorem.

**Theorem.** *Given positive integers  $s_1 = \overline{c_1 \cdots c_{k(b)}}$  and  $s_2 = \overline{d_1 \cdots d_{\ell(b)}}$  written in base  $b$ , there exist infinitely many positive integers  $n$  such that the base  $b$  representation of  $F_n$  starts with the digits of  $s_1$  and the base  $b$  representation of  $\phi(F_n)$  starts with the digits of  $s_2$ .*

We use the fact that with  $(\alpha, \beta) = ((1 + \sqrt{5})/2, (1 - \sqrt{5})/2)$  the Binet formula

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{holds for all} \quad n \geq 0.$$

For a positive real number  $x$  we write  $\log x$  for the natural logarithm of  $x$ , and  $\lfloor x \rfloor$ , respectively,  $\{x\}$ , for the integer part, respectively, fractional part of  $x$ .

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## 2. The proof

By replacing  $s_1$  with  $s_1 b^m$  for some positive integer  $m$ , if needed, whose effect is adding  $m$  zeros at the end of the base  $b$  representation of  $s_1$ , we may assume that  $s_1 > s_2$ . By replacing  $s_1, s_2$  by  $s_1 b^m, s_2 b^m$  for an arbitrary positive integer  $m$ , we may assume that the length of the base  $b$  representation of  $s_1$ , that is  $k$ , is as large as we wish. In Section 4 of [2], it is shown that  $\phi(F_n)/F_n$  is dense in  $[0, 1]$ . So, we take  $\varepsilon \in (0, 1/(15b^{2k}))$  and choose a positive integer  $a$  such that

$$\frac{\phi(F_a)}{F_a} \in \left( \frac{s_2}{s_1} + \varepsilon, \frac{s_2}{s_1} + 2\varepsilon \right). \quad (1)$$

Now we take any prime  $p > F_a$  and look at  $F_{ap}$ . Since  $p > F_a$ , it follows that

$$F_{ap} = F_a \left( \frac{F_{ap}}{F_a} \right),$$

and the two factors  $F_a$  and  $F_{ap}/F_a$  on the right above are coprime (indeed, the only common prime factor of these two numbers could be  $p$ , which is not the case since  $p > F_a$ ). Any prime factor  $q$  of  $F_{ap}/F_a$  is a *primitive prime factor* of  $F_{dp}$  for some divisor  $d$  of  $a$ . Recall that a prime number  $q$  is said to be a primitive prime factor of  $F_n$  if  $q$  divides  $F_n$ , but does not divide any  $F_m$  for  $1 \leq m < n$ . One of the properties of primitive prime factors  $q$  of  $F_n$  when  $n > 5$  is that  $q \equiv \pm 1 \pmod{n}$ . In particular, every prime factor  $q$  of  $F_{ap}/F_a$  is congruent to  $\pm 1 \pmod{p}$ .

Let  $q_1, \dots, q_t$  be all the prime factors of  $F_{ap}/F_a$ . Then

$$(2p-1)^t \leq q_1 \cdots q_t \leq \frac{F_{ap}}{F_a} \leq F_{ap} \leq \alpha^{ap}.$$

Thus,  $t = O(p/\log p)$ . Then

$$\begin{aligned} \frac{\phi(F_{ap})}{F_{ap}} &= \left( \frac{\phi(F_a)}{F_a} \right) \prod_{i=1}^t \left( 1 - \frac{1}{q_i} \right) \\ &= \frac{\phi(F_a)}{F_a} \exp \left( - \sum_{i=1}^t \frac{1}{q_i} + O \left( \sum_{q \geq q_1} \frac{1}{q^2} \right) \right) \\ &= \frac{\phi(F_a)}{F_a} \exp \left( O \left( \frac{t}{q_1} \right) \right) \\ &= \frac{\phi(F_a)}{F_a} \exp \left( O \left( \frac{1}{\log p} \right) \right) \end{aligned}$$

$$= \frac{\phi(F_a)}{F_a} \left( 1 + O\left(\frac{1}{\log p}\right) \right).$$

It implies that, if  $p > \exp(\kappa\varepsilon^{-1})$ , where  $\kappa > 0$  is some absolute constant, then

$$\frac{\phi(F_{ap})}{F_{ap}} \in \left( \frac{s_2}{s_1} + 0.5\varepsilon, \frac{s_2}{s_1} + 1.5\varepsilon \right). \quad (2)$$

We now follow Washington's argument [4] to prove that there exist infinitely many primes  $p$  such that the base  $b$  representation of  $F_{ap}$  starts with  $s_1$ . For this, it is enough to show that

$$F_{ap} = s_1 b^N + \zeta_{ap} \quad \text{for some integer} \quad 0 \leq \zeta_{ap} \leq b^N - 1. \quad (3)$$

Note that since  $q_1 \geq 2p - 1$ , it follows that if  $p$  is sufficiently large (say,  $p > b^k$ ), then  $F_{ap}$  cannot equal  $s_1 b^N$ , and in particular, if in the above formula (3) we have  $\zeta_{ap} \geq 0$ , then in fact  $\zeta_{ap} \geq 1$ . The above formula (3) yields

$$\alpha^{ap} = \sqrt{5} s_1 b^N + \sqrt{5} \zeta_{ap} + \beta^{ap} = \sqrt{5} s_1 b^N (1 + x_{ap}).$$

Since  $\zeta_{ap} \geq 1$ , it follows that  $\sqrt{5} \zeta_{ap} + \beta^{ap} > \sqrt{5} - 1 > 1$ , and

$$0 < x_{ap} = \frac{\sqrt{5} \zeta_{ap} + \beta^{ap}}{\sqrt{5} s_1 b^N}.$$

So, if  $x_{ap} \in (0, 1/b^k)$ , and  $p > b^k$  is sufficiently large, it then follows that  $\zeta_{ap} < b^N$ , which is what we want. Thus,

$$ap \log \alpha = \log(\sqrt{5} s_1) + N \log b + \log(1 + x_{ap}),$$

or

$$ap \frac{\log \alpha}{\log b} - N - \left\lfloor \frac{\log(\sqrt{5} s_1)}{\log b} \right\rfloor = \left\{ \frac{\log(\sqrt{5} s_1)}{\log b} \right\} + \frac{\log(1 + x_{ap})}{\log b}. \quad (4)$$

Observe that  $\log(\sqrt{5} s_1)/\log b$  is never an integer. Assume that  $k$  is sufficiently large such that

$$\frac{1}{b^k \log b} < 1 - \left\{ \frac{\log(\sqrt{5} s_1)}{\log b} \right\}.$$

Then putting

$$\delta = \frac{\log(1 + 1/b^k)}{\log b},$$

we see that a relation like (4) with  $x_{ap} \in (0, 1/b^k)$  holds provided that

$$\left\{ p \left( \frac{a \log \alpha}{\log b} \right) \right\} \in \left( \left\{ \frac{\log(\sqrt{5} s_1)}{\log b} \right\}, \left\{ \frac{\log(\sqrt{5} s_1)}{\log b} \right\} + \delta \right). \quad (5)$$

The number  $\gamma = a \log \alpha / \log b$  is irrational. By a result of Vinogradov [3], the sequence of fractional parts  $\{p\gamma\}_{p \text{ prime}}$  is uniformly distributed. In particular, containment (5) holds for a positive proportion of primes  $p$ , and therefore certainly for infinitely many of them. So, indeed relation (3) holds. Relation (2) now shows that

$$\phi(F_{ap}) = s_2 b^N + \theta,$$

where

$$\theta \in \left( \zeta_{ap} \left( \frac{s_2}{s_1} + 0.5\varepsilon \right) + 0.5\varepsilon b^N, \zeta_{ap} \left( \frac{s_2}{s_1} + 1.5\varepsilon \right) + 1.5\varepsilon s_1 b^N \right).$$

Since  $\varepsilon < 1/(15b^{2k})$ , the above upper bound is

$$\begin{aligned} \zeta_{ap} \left( \frac{s_2}{s_1} + 1.5\varepsilon \right) + 1.5\varepsilon s_1 b^N &< (b^N - 1) \left( \frac{b^k - 1}{b^k} + \frac{0.1}{b^k} \right) + \frac{0.1(b^k - 1)}{b^{2k}} b^N \\ &< b^N - 1, \end{aligned}$$

where the last inequality above is implied by

$$\frac{1}{9} < \frac{b^N - 1}{b^N},$$

which holds true for all  $b \geq 2$  and  $N \geq 1$ . This completes the proof of the theorem.

### 3. Comments

It was shown in [1] that with  $\sigma(m)$  being the sum of divisors of the positive integer  $m$ , the ratio  $\sigma(F_n)/F_n$  is dense in  $[1, \infty)$ . The present method now shows that there are infinitely many positive integers  $n$  such that the base  $b$  representation of  $F_n$  starts with the digits of  $s_1$  and the base  $b$  representation of  $\sigma(F_n)$  starts with the digits of  $s_2$ . Also, one may replace the Fibonacci sequence  $F_n$  in the above statements with some other sequence  $u_n$  for which it has been proved that  $\phi(u_n)/u_n$  and  $u_n/\sigma(u_n)$ , respectively, are dense in  $[0, 1]$ . For example, one can take  $u_n = 2^n - 1$  (see [1]) and the main result of this paper still holds provided that  $b$  is not a power of 2. We give no further details.

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